Advances and Applications in Mechanical Engineering and Technology
Volume 1, Number 2, 2010, Pages 133-146

RESEARCH ON PLANE EQUIVALENT TRUSS FINITE ELEMENT MODEL OF REINFORCED CONCRETE

WU FANGBO, WEN JUN and ZHOU XUHONG

College of Civil Engineering
Hunan University
Changsha, 410082
P. R. China
e-mail: wfbprof@163.com

Abstract

According to the basic theory of reinforced concrete finite element method, this paper proposes a concise element, equivalent truss element. To the existing reinforced concrete finite element models, element stiffness matrices of equivalent truss integrated model, equivalent truss separated model, and equivalent truss combined model are derived. Meanwhile, the three models are compared ulteriorly.

1. Introduction

Reinforced concrete finite element method is a powerful tool for studying the behavior of reinforced concrete structure. In reinforced concrete finite element analysis, nonlinear characteristics of concrete and steel can be considered in the computation module. Bonding between steel and concrete can also be considered and simulated; beam-column joints and boundary conditions can also be simulated in a certain extent; it will provide plentiful structural reaction information; it is helpful to the improvement of test, and it can even replace partial test [3].

Keywords and phrases: reinforced concrete finite element analysis, equivalent truss element, integrated model, separated model, combined model.

Received April 2, 2010

At home and abroad, triangle element and rectangular element are commonly used in plane finite element analysis. In nonlinear analysis, calculation precision lies mostly on the selection of constitutive relationship of concrete and steel. The nonlinear constitutive relationship of concrete has many different forms. It has being a bottleneck for reinforced concrete finite element analysis to select, which constitutive relationship. Meanwhile, the crack of concrete has enormous influence on the mechanical performance of reinforced concrete structure members. Now, the commonly used crack models are smeared crack model, discrete crack model, and embedded crack model. It has also being a bottleneck problem for reinforced concrete finite element analysis to select, which crack model. In order to solve the above problems, based on the mechanical performance of reinforced concrete structure members and the thought of finite element method, a new type of element, namely, equivalent truss model [5], is proposed to replace the plane stress element. In general, the elements to simulate the reinforced concrete structures can be divided into three kinds, that is, integrated model, separated model, and combined model. However, only equivalent truss integrated model was preliminarily studied. In this paper, element stiffness matrices of equivalent truss integrated model, equivalent truss separated model, and equivalent truss combined model are derived systematically. Meanwhile, the three models are compared ulteriorly.

2. Element Stiffness Matrix of Equivalent Truss Model

Equivalent truss model is equivalent with the plane stress element. Equivalent principle is stiffness equivalent, that is, under the same load, the equivalent model will have the same displacement with the plane stress element. Equivalent truss model consists of six bars. The bars are all two-force bars. Four peripheral bars are connected end to end to be a square. Two inner bars are not connected rigidly, but are crossed and unattached with each other. Equivalent truss element has the same peripheral geometry size with the plane stress element, as shown in

Figure 1. It is assumed that the length of the two bars parallel to x axis is dx, the length of the two bars parallel to y axis is dy, and the thickness parallel to z axis is t. In a square element, $dx = dy = L$.

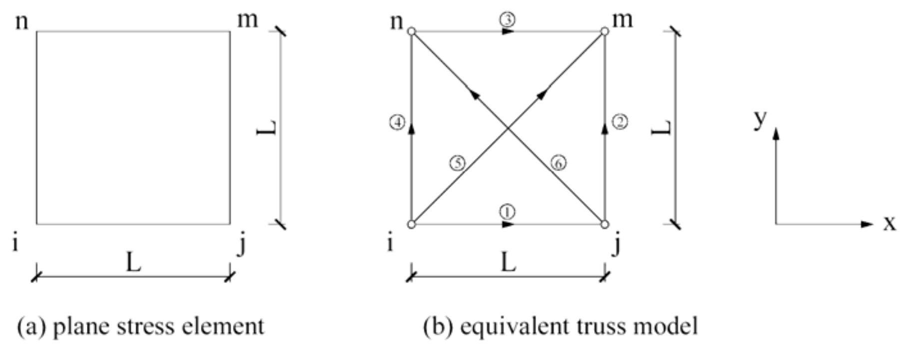

Figure 1. Geometry size of equivalent truss model and plane stress element.

Based on the finite element method basic ideology, element stiffness matrix of equivalent truss model is derived as following. From plane stress element, according to static equivalent principle, equivalent node loads are calculated. Then equivalent stiffness of each bar is calculated based on principle of the stiffness is equivalent, and stiffness contribution matrix of each bar is calculated by treating each bar as a plane beam element. Element stiffness matrix of equivalent truss model will be integrated of each bar stiffness contribution matrix according to the relationship of each bar node displacement and element node displacement.

2.1. Load equivalent analysis

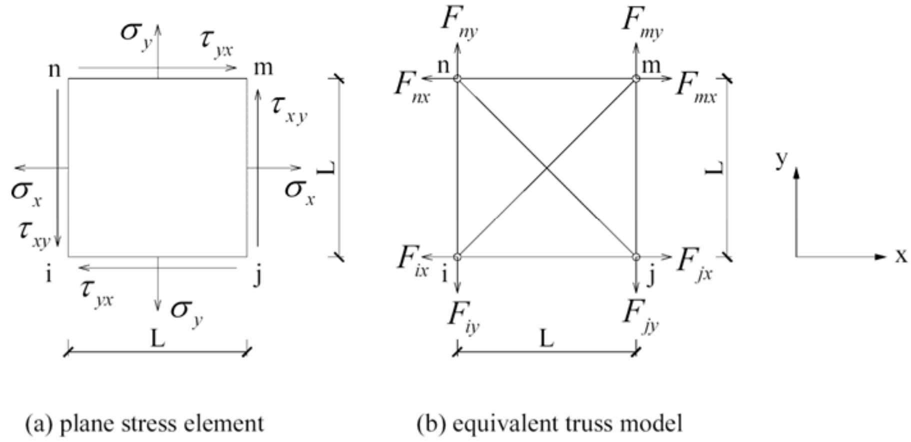

Figure 2. Load equivalent relationship of equivalent truss element and plane stress element.

Because the plane stress element is micro, it can be presumed that normal stress and shearing stress distribute uniformly. Referring to Figure 2, it is assumed that normal stress is σ_x, shearing stress is τ_{xy} in the plane, which is normal to *x* axis; and it is assumed that normal stress is σ_y, shearing stress is τ_{yx} in the plane, which is normal to *y* axis. According to the shear stress mutual equal theory, it can be known that $\tau_{xy} = \tau_{yx} = \tau$. Using static equivalent principle [6], equivalent nodal forces of equivalent truss model can be calculated by Equation 1.

$$\begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \\ F_{mx} \\ F_{my} \\ F_{nx} \\ F_{ny} \end{bmatrix} = ht \begin{bmatrix} \sigma_x / 2 + \tau / 2 \\ \sigma_y / 2 + \tau / 2 \\ \sigma_x / 2 - \tau / 2 \\ \sigma_y / 2 - \tau / 2 \\ \sigma_x / 2 + \tau / 2 \\ \sigma_y / 2 + \tau / 2 \\ \sigma_x / 2 - \tau / 2 \\ \sigma_y / 2 - \tau / 2 \end{bmatrix}. \tag{1}$$

2.2. Displacement equivalent analysis

Under the load as shown in Figure 2, the plane stress element and equivalent truss model will produce distortion that includes changes of element's side length and angle between two neighboring sides. It is assigned that side length change parallel to x axis is Δ_1, side length change parallel to y axis is Δ_2, and change of angle between two neighboring sides is Δ_3 in plane stress element; and side length change parallel to x axis is Δ_1', side length change parallel to y axis is Δ_2', and change of angle between two neighboring sides is Δ_3' in equivalent truss model as shown in Figure 3.

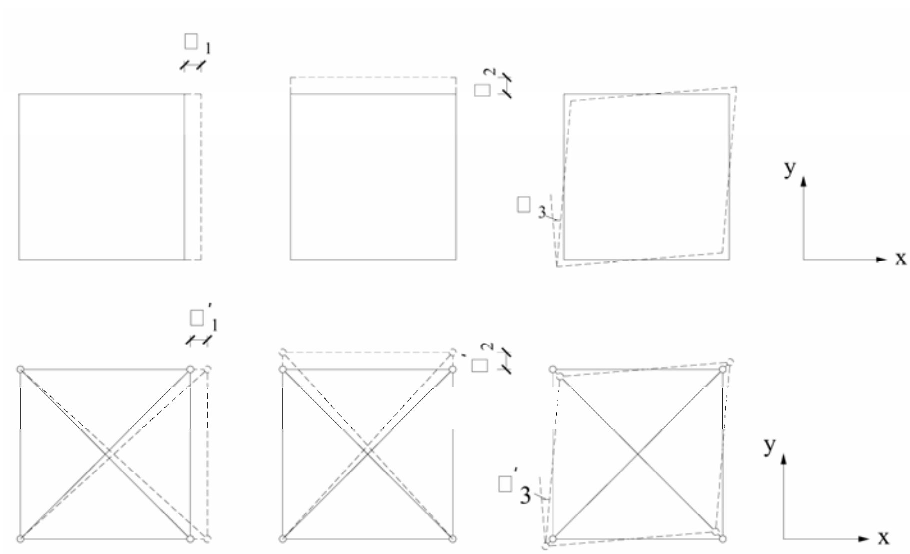

Figure 3. Deformation of equivalent truss element and plane stress element.

Applying Hook's law, Δ_1, Δ_2, and Δ_3 can be calculated by Equation 2.

$$\Delta_1 = \frac{\sigma_x L}{E} - \upsilon \frac{\sigma_y L}{E},$$

$$\Delta_2 = \frac{\sigma_y L}{E} - \upsilon \frac{\sigma_x L}{E}, \tag{2}$$

$$\Delta_3 = \frac{\tau}{G},$$

where E is modulus of elasticity, G is shear modulus, and v is Poisson's ratio.

Applying general method of solving displacement problems, Δ_1', Δ_2', and Δ_3' can be calculated by Equation 3 based on virtual work principle.

$$\Delta_1' = \frac{\sigma_x L^2 t}{2} \frac{2\sqrt{2}}{2\sqrt{2}\alpha + \gamma} - v\Delta_2',$$

$$\Delta_2' = \frac{\sigma_y L^2 t}{2} \frac{2\sqrt{2}}{2\sqrt{2}\beta + \gamma} - v\Delta_1', \tag{3}$$

$$\Delta_3' = \frac{\sqrt{2}\tau L t}{\gamma},$$

where $\alpha, \beta,$ and γ are the axial stiffness of the bars of the equivalent truss element in x-direction, y-direction, and diagonal direction, respectively.

The equivalent principle can be expressed as Equation 4.

$$\Delta_1' = \Delta_1,$$

$$\Delta_2' = \Delta_2, \tag{4}$$

$$\Delta_3' = \Delta_3.$$

From Equations 2, 3, and 4, the axial stiffness ($\alpha, \beta,$ and γ) of the bars of the equivalent truss model can be expressed as:

$$\alpha = \beta = \frac{1 + 2v}{2(1 + v)} ELt,$$

$$\gamma = \frac{1}{2(1 + v)} \sqrt{2} ELt. \tag{5}$$

Equivalent truss model is comprised of six two-force bars. Firstly, the contribution matrix of each bar is calculated by treating each bar as a plane beam element, and then according to the relationship between each bar node displacement and element node displacement, the six contribution matrices are integrated to element stiffness matrix of the equivalent truss model as shown in Equation 6.

$$[K]=\frac{1}{L}\begin{bmatrix} \alpha+\frac{\gamma}{2\sqrt{2}} & & & & & & & \\ \frac{\gamma}{2\sqrt{2}} & \beta+\frac{\gamma}{2\sqrt{2}} & & & & & & \\ -\alpha & 0 & \alpha+\frac{\gamma}{2\sqrt{2}} & & & \text{symmetrical} & & \\ 0 & 0 & -\frac{\gamma}{2\sqrt{2}} & \beta+\frac{\gamma}{2\sqrt{2}} & & & & \\ -\frac{\gamma}{2\sqrt{2}} & -\frac{\gamma}{2\sqrt{2}} & 0 & 0 & \alpha+\frac{\gamma}{2\sqrt{2}} & & & \\ -\frac{\gamma}{2\sqrt{2}} & -\frac{\gamma}{2\sqrt{2}} & 0 & -\beta & \frac{\gamma}{2\sqrt{2}} & \beta+\frac{\gamma}{2\sqrt{2}} & & \\ 0 & 0 & -\frac{\gamma}{2\sqrt{2}} & \frac{\gamma}{2\sqrt{2}} & -\alpha & 0 & \alpha+\frac{\gamma}{2\sqrt{2}} & \\ 0 & -\beta & \frac{\gamma}{2\sqrt{2}} & -\frac{\gamma}{2\sqrt{2}} & 0 & 0 & -\frac{\gamma}{2\sqrt{2}} & \beta+\frac{\gamma}{2\sqrt{2}} \end{bmatrix}. \tag{6}$$

3. Plane Element Types

Basic principle and methods of reinforced concrete finite element analysis are the same as general solid mechanics finite element analysis. But, because reinforced concrete structure is composed of two different materials, concrete and steel, so operation process of reinforced concrete finite element analysis is quite different from general solid mechanics finite element analysis. In reinforced concrete finite element analysis, the relationship between concrete and steel must be considered. In general, the elements to simulate the reinforced concrete structures can be divided into three kinds, that is, integrated model, separated model, and combined model [1], [3], [4]. This three models for equivalent truss model are built in this paper, and general expressions of element stiffness matrix are derived.

3.1. Integrated model

In integrated model, steel is dispersed equally in the whole element, and the element is treated as a homogeneous material. Element stiffness matrix of integrated model colligates the two materials, steel and concrete. Element stiffness matrix for equivalent truss integrated model is derived as following.

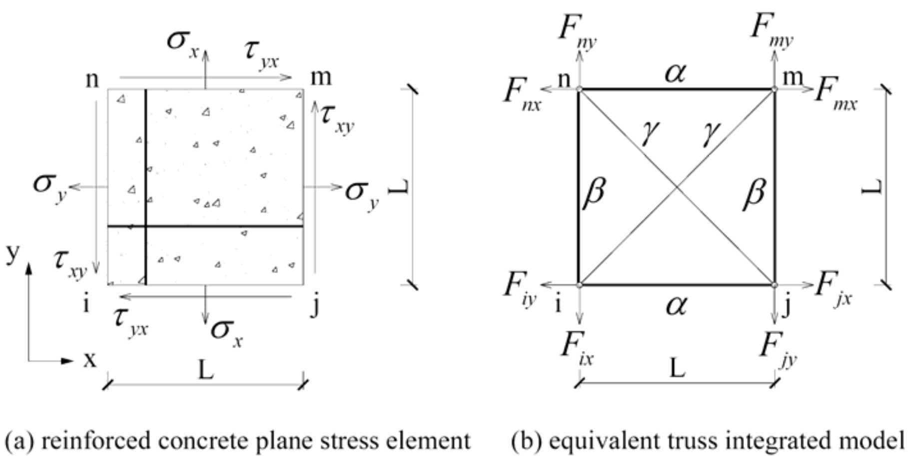

Figure 4. Equivalent relationship of plane stress element and equivalent truss integrated model.

Referring to Figure 4, in equivalent truss integrated model, steel in x-direction is assigned in average to two truss bars parallel to x axis; steel in y-direction is assigned in average to two truss bars parallel to y axis. Four peripheral bars are comprised of two materials, steel and concrete. Basing basic principle of mechanics of materials and structural mechanics [7], the axial stiffness of the bars (α, β, and γ) of the equivalent truss integrated model can be calculated by Equation 7. Element stiffness matrix can be calculated by substituting Equation 7 for Equation 6.

$$\alpha = \frac{(E_{cx} + \rho_{sx}E_{sx} - G)Lt}{2},$$
$$\beta = \frac{(E_{cy} + \rho_{sy}E_{sy} - G)Lt}{2}, \tag{7}$$
$$\gamma = \sqrt{2}LtG,$$

where L is side length of equivalent truss integrated model; t is thickness parallel to z axis; E_{cx}, E_{cy}, and G are x-direction tangent module of concrete, y-direction tangent module of concrete, and shear module of concrete, respectively; E_{sx} and E_{sy} are x-direction and y-direction tangent module of steel, respectively; ρ_{sx} and ρ_{sy} are x-direction and y-direction reinforcement ratio, respectively.

3.2. Separated model

In separated model, steel and concrete are divided into a finite number of small elements, respectively. Appropriate element forms for steel and concrete need to be selected according to mechanical performance of steel and concrete. Equivalent truss separated model is shown as Figure 5. In equivalent truss separated model, equivalent truss model, which is equivalent with the plane stress element is adopted to simulate concrete, and its element stiffness matrix can be calculated by Equation 6. Because steel is a slender material relative to concrete, plane beam element [2] is adopted to simulate steel and its element stiffness matrix can be calculated by Equation 8. In order to reveal the micro-mechanism of the interaction between steel and concrete, quadrilateral joint element [1], [3], [4] as shown in Figure 6 is adopted to simulate the bond and relative slip between steel and concrete, and its element stiffness matrix can be calculated by Equation 9.

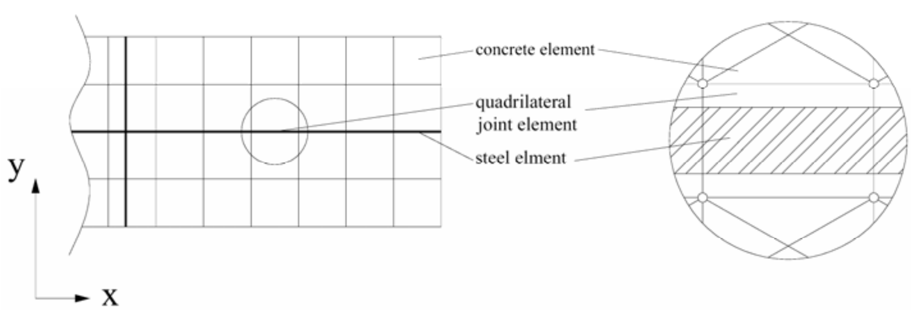

Figure 5. Equivalent truss separated model.

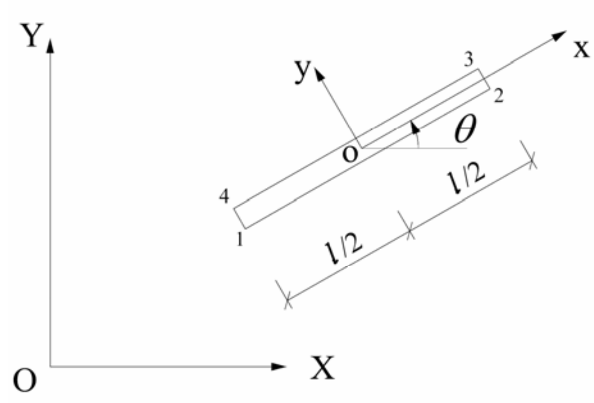

Figure 6. Quadrilateral joint element.

$$[k_s] = \frac{EA}{L}\begin{bmatrix} C^2 & & & \\ CS & S^2 & \text{symmetrical} & \\ -C^2 & -CS & C^2 & \\ -CS & -S^2 & CS & S^2 \end{bmatrix}, \tag{8}$$

where E, A, and L are modulus of elasticity, cross-sectional area, and length of steel bar, respectively; $C = \cos\varphi$, $S = \sin\varphi$, φ is the included angle between steel bar and x axis.

$$[k] = \frac{l}{6}\begin{bmatrix} 2a & & & & & & & \\ 2b & 2c & & & & & & \\ a & b & 2a & & & \text{symmetrical} & & \\ b & c & 2b & 2c & & & & \\ -a & -b & -2a & -2b & 2a & & & \\ -b & -c & -2b & -2c & 2b & 2c & & \\ -2a & -2b & -a & -b & a & b & 2a & \\ -2b & -2c & -b & -c & b & c & 2b & 2c \end{bmatrix}, \tag{9}$$

where

$$a = k'_x \cos^2\theta + k'_y \sin^2\theta,$$

$$b = (k'_x - k'_y)\sin\theta\cos\theta,$$

$$c = k_x' \sin^2 \theta + k_y' \cos^2 \theta, \tag{10}$$

where k_x' and k_y' are tangential and normal stiffness coefficient of quadrilateral joint element, which can be determined by testing; θ is the included angle between local coordinate axis *ox* and global coordinate axis *OX* as shown in Figure 6.

3.3. Combined model

Combined model is between integrated model and separated model. In the combined model, the function of tendon (*s*) is considered by the stiffness matrix with rod element, then add it into the stiffness matrix of concrete by a rational way. So that, it can express the function of tendon (*s*) and constitution of RC member, without increasing of degree of freedom. During the element analysis, firstly, stiffness contribution matrices of steel and concrete are calculated, respectively, and then they are integrated into a multiple element stiffness matrix.

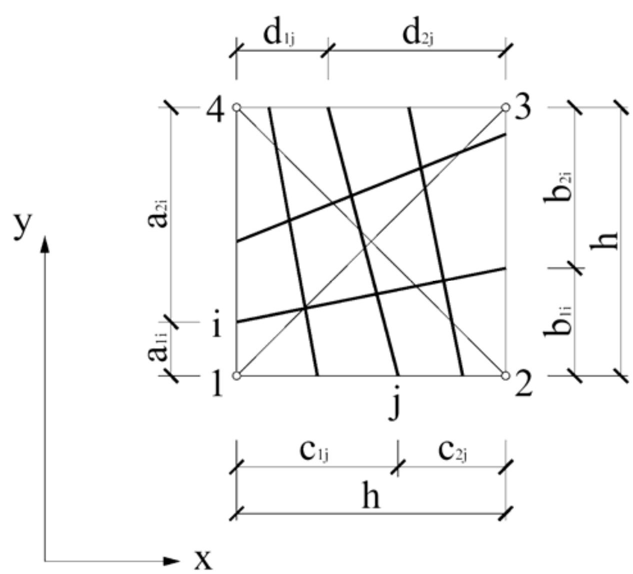

Figure 7. Equivalent truss combined model.

In equivalent truss combined model, stiffness contribution matrix of concrete can be calculated by Equation 6. In order to obtain the stiffness contribution matrix of steel, it is assumed that there are m longitudinal steel bars and n transversal steel bars as shown in Figure 7. Employing the virtual displacement principle, coordinate transmitting matrix of the i-th longitudinal steel bar can be expressed as Equation 11, which build the relationship of steel bar node displacement and element node displacement.

$$[R_i] = \begin{bmatrix} a_{2i}/h & 0 & 0 & 0 & 0 & 0 & a_{1i}/h & 0 \\ 0 & a_{2i}/h & 0 & 0 & 0 & 0 & 0 & a_{1i}/h \\ 0 & 0 & b_{2i}/h & 0 & b_{1i}/h & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{2i}/h & 0 & b_{1i}/h & 0 & 0 \end{bmatrix}. \tag{11}$$

Simultaneously, coordinate transmitting matrix of the j-th transversal steel bar can be expressed as Equation 12.

$$[R_j] = \begin{bmatrix} c_{2j}/h & 0 & c_{1j}/h & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{2j}/h & 0 & c_{1j}/h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{1j}/h & 0 & d_{2j}/h & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{1j}/h & 0 & d_{2j}/h \end{bmatrix}. \tag{12}$$

Corresponding stiffness contribution matrix can be calculated by Equation 13.

$$k_{shi} = R_i^T \overline{k_{shi}} R_i,$$

$$k_{svj} = R_j^T \overline{k_{svj}} R_j, \tag{13}$$

where $\overline{k_{shi}}$ is stiffness matrix of the i-th longitudinal steel bar; $\overline{k_{svj}}$ is stiffness matrix of the j-th transversal steel bar, which can be calculated as plane beam element by Equation 8.

Stiffness contribution matrix of steel is expressed as Equation 14.

$$k_s = \sum_{i=1}^{m} k_{shi} + \sum_{j=1}^{n} k_{svj}. \tag{14}$$

Element stiffness matrix of equivalent truss combined model is expressed as Equation 15.

$$k_{sc} = k_c + k_s. \tag{15}$$

Equivalent truss model is a new unit form, especially adequate for reinforced concrete finite element analysis. The major advantage of this model is the simplified finite element model without use of inserting functions for displacements or high order differential of inserting functions. Secondly, in non-linear analysis, selection of appropriate material constitutive relationship in traditional finite element approaches is often critical and some times problematic to the accuracy of the results. Employing equivalent truss model, because the composed bars are all two-force bars, uniaxial stress-strain relationship can be adopted to substitute material constitutive relationship under multi-axial stress states. Fortunately, research on uniaxial stress-strain relationship is relatively mature. Thirdly, equivalent truss model provides a new cracking treatment method. Position of cracking members can be recorded randomly without presupposed cracking position as in discrete cracking model or smeared cracking model.

4. Conclusion

In this paper, element stiffness matrices of equivalent truss integrated model, equivalent truss separated model, and equivalent truss combined model are derived systematically. Meanwhile, this three models are compared ulteriorly. In equivalent truss separated model, quadrilateral joint element is adopted to reveal the micro-mechanism of the interaction between steel and concrete, which makes it superior to equivalent truss integrated model and combined model. However, for the equivalent truss separated model, concrete element stiffness matrix, steel element stiffness matrix, and quadrilateral joint element stiffness matrix must be calculated, respectively, so number of element is great results higher need of computer capacity and velocity. Because of the complex of structure discretization, it is still difficult to apply in real project. In equivalent truss integrated model and combined model, element stiffness matrix integrates the two materials, steel and concrete. For the equivalent truss integrated model, steel is dispersed equally in the whole element weakening the practical function of steel. Equivalent truss combined model can reflect the precision function of steel and it is recommended.

References

[1] Jianjing Jiang, Xinzheng Lu and Lieping Ye, Reinforced Concrete Structure Finite Element Analysis, Tsinghua University Publishing House, Beijing, China, 2005.

[2] D. Logan, A First Course in the Finite Element Method, PWS Publishing Company, ITP, 1993.

[3] Xiling Lu, Guofang Jin and Xiaohan Wu, Theory and Application of Finite Element Method in Non-linear Analysis of Reinforced Concrete Structure, Tingji University Publishing, Shanghai, China, 2002.

[4] Jumin Shen, Chuanzhi Wang and Jianjing Jiang, Finite Element Method Application in Reinforced Concrete Structure and the Analysis of Shell Elements under Extreme State, TSinghua University Publishing House, Beijing, China, 1993.

[5] Fangbo Wu, Xianli Ding, Xuhong Zhou and Shaoyao He, Study on the nonlinear analysis of reinforced concrete structures by equivalent plane truss element method, Journal of Building Structure 26(5) (2005), 112-117.

[6] Zhilun Xu, Simple Theory of Elasticity Mechanics, Higher Education Publishing House, Beijing, China, 2003.

[7] Fukang Yang and Jiabao Li, Structural Mechanics, Higher Education Publishing House, Beijing, China, 2001.

■